

(t, s) -racks and their link invariants

Jessica Cenicerros*

Sam Nelson†

Abstract

A (t, s) -rack is a rack structure defined on a module over the ring $\check{\Lambda} = \mathbb{Z}[t^{\pm 1}, s]/(s^2 - (1 - t)s)$. We identify necessary and sufficient conditions for two (t, s) -racks to be isomorphic. We define enhancements of the rack counting invariant using the structure of (t, s) -racks and give some computations and examples. As an application, we use these enhanced invariants to obtain obstructions to knot ordering.

KEYWORDS: Finite racks, (t, s) -racks, Alexander quandles, link invariants, enhancements of counting invariants

2010 MSC: 57M27, 57M25

1 Introduction

Introduced in [7], the *fundamental rack* of a framed link is a complete invariant of unsplit framed links in S^3 up to homeomorphism of S^3 . Counting homomorphisms from a fundamental rack into a finite rack X yields an invariant of framed isotopy. In [10] it is shown that this counting invariant is periodic with respect to framings modulo an integer $N(X)$ known as the *rack rank* of X , and that summing these counting invariants over a complete period of framings modulo $N(X)$ yields an invariant of ambient isotopy.

In this paper we study a type of rack structure on modules over the ring $\check{\Lambda} = \mathbb{Z}[t^{\pm 1}, s]/(s^2 - (1 - t)s)$ known as (t, s) -racks and the counting invariants they define. We obtain a result specifying necessary and sufficient conditions for two (t, s) -racks to be isomorphic, similar to results for Alexander quandles and Alexander biquandles in [9] and [11] respectively. We are able to exploit the module structure of these racks to enhance the rack counting invariant, yielding stronger invariants which specialize to the unenhanced counting invariant.

The paper is organized as follows. In section 2 we review the basics of racks and the rack counting invariant. In section 3 we introduce (t, s) -racks and provide necessary and sufficient conditions for two (t, s) -racks to be isomorphic. In section 4 we define the new enhanced invariants, give examples and provide an application to knot ordering. In section 5 we collect questions for future research.

2 Rack basics

Definition 1 A *rack* is a set X with two binary operations $\triangleright, \triangleright^{-1}$ satisfying for all $x, y \in X$

- (i) $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ and
- (ii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

It follows from (i) and (ii) (see [10]) that the *kink map* $\pi : X \rightarrow X$ defined by $\pi(x) = x \triangleright x$ is a bijection. For every element $x \in X$, the *rack rank* of x is the smallest integer $N(x) \geq 1$ such that $\pi^{N(x)}(x) = x$ or ∞ if no such $N(x)$ exists, and the *rack rank* of X is smallest positive integer $N(X)$ such that $\pi^{N(X)}(x) = x$ for all $x \in X$, or ∞ if no such N exists. A rack with rack rank $N = 1$ is a *quandle*. We will denote the kink map in X by $\pi_X : X \rightarrow X$ when necessary to distinguish it from the kink maps of other racks.

*Email: jcenicerros11@students.claremontmckenna.edu

†Email: knots@esotericka.org

We have the following standard result (or see [10]):

Lemma 1 *If X is a finite rack, then $N(x) \neq \infty$ for all $x \in X$ and $N(X) = \text{lcm}\{N(x) \mid x \in X\}$.*

Proof. Let X be a finite rack and consider the map $f_x : X \rightarrow X$ defined by $f_x(y) = y \triangleright x$. For each x , f_x is an element of the symmetric group $S_{|X|}$ and hence has finite order equal to $N(x)$. Since $N(x)$ must divide $N(X)$, for all $x \in X$, we must have $N(X) = \text{lcm}\{N(x) \mid x \in X\}$. \square

As with other algebraic structures, we have some useful standard concepts:

Definition 2 Let X and Y be racks.

- A *subrack* of X is a subset $S \subset X$ which is itself a rack under the rack operations $\triangleright, \triangleright^{-1}$ inherited from X . For $S \subset X$ to be a subrack, it is sufficient for S to be closed under the rack operations \triangleright and \triangleright^{-1} . If the rack rank of X is finite, then closure under \triangleright implies closure under \triangleright^{-1} .
- A *rack homomorphism* is a map $f : X \rightarrow Y$ satisfying for all $x, y \in X$

$$f(x \triangleright y) = f(x) \triangleright f(y) \quad \text{and} \quad f(x \triangleright^{-1} y) = f(x) \triangleright^{-1} f(y).$$

- The *image* of a homomorphism $f : X \rightarrow Y$ is the set $\text{Im}(f) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$; it is straightforward to show that $\text{Im}(f)$ is a subrack of Y .

We will find the following observations useful in section 2.

Lemma 2 *Let $f : X \rightarrow Y$ be a rack homomorphism. Then for any $x \in X$, the rack rank of $f(x)$ divides the rack rank of x .*

Proof. Let $f : X \rightarrow Y$ be a rack homomorphism. Then for any $x \in X$ we have

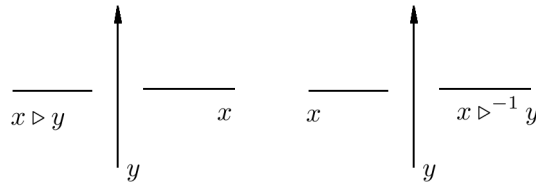
$$f(\pi_X(x)) = f(x \triangleright x) = f(x) \triangleright f(x) = \pi_Y(f(x)).$$

Then if $\pi_X^N(x) = x$, we have $\pi_Y^N(f(x)) = f(\pi_X^N(x)) = f(x)$ so $\pi_Y^{N(X)}(f(x)) = f(x)$, and $N(f(x)) \mid N(x)$ as required. \square

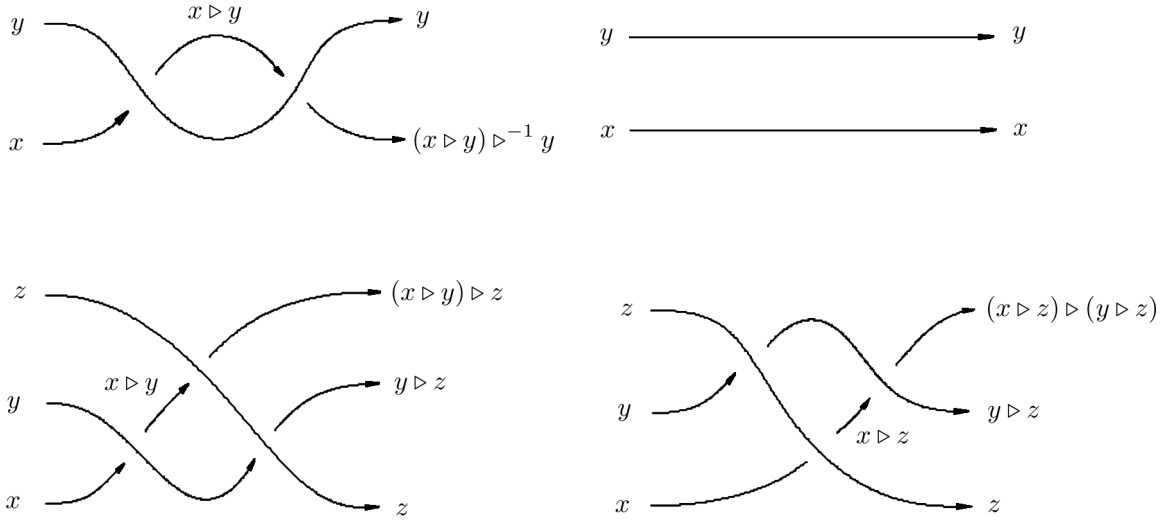
Corollary 3 *If $f : X \rightarrow Y$ is an isomorphism of racks then $\pi_X = f^{-1}\pi_Y f$, i.e. the kink maps of X and Y are conjugate.*

Corollary 4 *If two racks X and Y are isomorphic, then the rack ranks of X and Y are equal.*

The rack axioms come from the blackboard-framed oriented Reidemeister moves where we interpret $x \triangleright y$ as the arc resulting from x crossing under y from right to left with respect to the orientation of the overcrossing strand and $x \triangleright^{-1} y$ as crossing under from left to right [7].



Axiom (i) comes from Reidemeister move II, axiom (ii) comes from the oriented Reidemeister III move with all positive crossings.

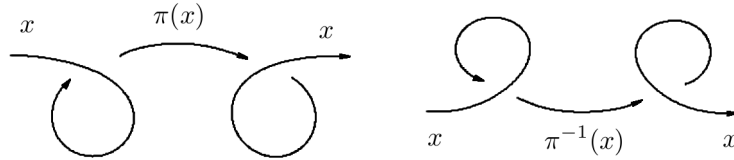


The other oriented Reidemeister III moves follow from the listed moves, with corresponding rack equations such as

$$(x \triangleright y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright (y \triangleright^{-1} z).$$

See [7] for more.

The blackboard-framed oriented Reidemeister I moves do not impose any additional axioms, but provide a visual interpretation of the kink map: $\pi(x)$ is the result of x going through a positive-writhe kink, and $\pi^{-1}(x)$ is the result of going through a negative-writhe kink.



Standard examples of rack structures include:

- *Constant action racks:* A set X with a bijection $\sigma : X \rightarrow X$ is a rack with $x \triangleright y = \sigma(x)$,
- *Conjugation racks:* A group G is a rack with $x \triangleright y = y^{-n}xy^n$ for each $n \in \mathbb{Z}$,
- *Coxeter racks:* The subset $S \subset V$ of an \mathbb{F} -vector space V which is non-degenerate with respect to a symmetric bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{F}$ is a rack with

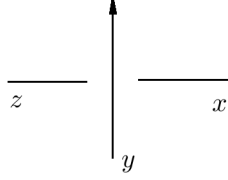
$$\mathbf{x} \triangleright \mathbf{y} = \mathbf{y} - 2 \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} \mathbf{x},$$

- *Fundamental rack of a link L :* Let L be a blackboard-framed oriented link diagram and let G be a set of generators corresponding bijectively with the set of arcs in L . Define the set $W(G)$ of *rack words in G* recursively by the rules

- $g \in G \Rightarrow g \in W(G)$ and

- $g, h \in W(G) \Rightarrow g \triangleright h \in W(G)$ and $g \triangleright^{-1} h \in W(G)$.

Let \sim be the equivalence relation on $W(G)$ generated by the rack axioms (e.g., $(x \triangleright y) \triangleright z \sim (x \triangleright z) \triangleright (y \triangleright z)$ etc.) together with the crossing relations in L , i.e., for every crossing

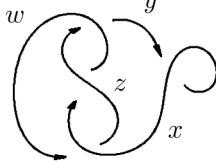


we obtain a relation $z \sim x \triangleright y$ or, equivalently, $x \sim z \triangleright^{-1} y$. Then the set $FR(L) = W(G)/\sim$ of equivalence classes in $W(G)$ modulo the equivalence relation \sim is a rack under the operations

$$[x] \triangleright [y] = [x \triangleright y] \quad \text{and} \quad [x] \triangleright^{-1} [y] = [x \triangleright^{-1} y]$$

where $[x]$ is the equivalence class of x in $W(G)/\sim$. In particular, the fundamental racks of any two oriented blackboard-framed link diagrams related by blackboard framed Reidemeister moves are isomorphic.

This last example is especially important; in [7] it is shown that the Fundamental Rack of a framed link is a complete invariant for unsplit framed links, up to homeomorphism of the ambient space S^3 . For example, the blackboard-framed trefoil below has fundamental rack with the listed presentation:



$$FR(K) = \langle x, y, z, w \mid y \triangleright x = x, z \triangleright x = w, w \triangleright z = x, y \triangleright w = z \rangle.$$

A finite rack $X = \{x_1, \dots, x_n\}$ can be expressed in an algebra-agnostic way using a *rack matrix* which encodes the operation table of X . Specifically, the entry in row i column j of M_X is k where $x_i \triangleright x_j = x_k$. For example, the constant action rack on $X = \{x_1, x_2, x_3\}$ with $\sigma = (123)$ has rack matrix

$$M_X = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

The kink map $\pi(x)$ is the permutation along the diagonal of the rack matrix; in this example, $\pi = (123)$ and $N = 3$.

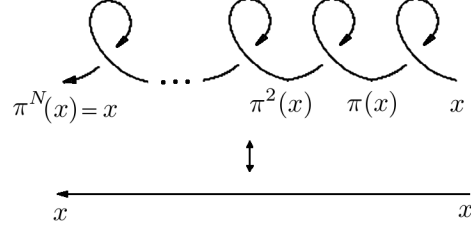
By construction, any labeling of a diagram D of a blackboard-framed oriented link L with elements of a rack X satisfying the crossing condition at every crossing corresponds to a unique such labeling on any diagram obtained from D by a blackboard-framed Reidemeister move. More abstractly, such a labeling is an assignment of an image $f(g) = x_i \in X$ to each generator g of $FR(L)$, and satisfaction of the crossing conditions says that f defines a unique homomorphism of racks $f : FR(L) \rightarrow X$. The set of such labelings or homomorphisms is an invariant of blackboard framed isotopy denoted

$$\text{Hom}(FR(L), X) = \{f : FR(L) \rightarrow X \mid f(x \triangleright y) = f(x) \triangleright f(y)\}.$$

The cardinality $|\text{Hom}(FR(L), X)|$ is a numerical invariant known as the *basic counting invariant*.

For a finite rack X , the rack rank N is always finite – indeed, N is the exponent or order of the kink map $\pi : X \rightarrow X$ considered as an element of the symmetric group $S_{|X|}$. The finiteness of N for a rack X

implies that the basic counting invariants are periodic in the writhe w of each component of L with period N – in particular, the basic counting invariant is preserved by N -phone cord moves:

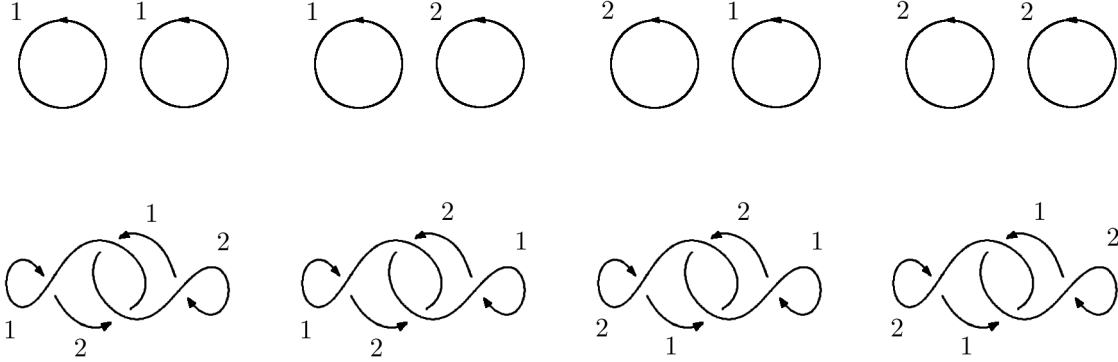


Thus, if we let $W = (\mathbb{Z}_N)^c$ where c is the number of components of L and let $D(L, \mathbf{w})$ be a diagram of L with framing vector \mathbf{w} for a fixed ordering of the components of L , then the number

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in W} |\text{Hom}(FR(D(L, \mathbf{w})), X)|$$

is an invariant of the unframed link L called the *integral rack counting invariant*. In the special case that X is a quandle, i.e. $N = 1$, this is just the basic counting invariant $|\text{Hom}(FR(L), X)|$.

Example 1 The constant action rack X with $\sigma = (12)$ has rack rank 2. We can interpret the rack operation as a labeling rule which says that each time an arc goes under a crossing, the label switches from 1 to 2 or from 2 to 1. Since we have $N = 2$, to compute $\Phi_X^{\mathbb{Z}}(L)$ we need a complete set of framing vectors over $(\mathbb{Z}_2)^c$. For example, the Hopf link H_2 and the 2-component unlink U_2 both have four labelings by X , but they occur in different framings, with the only valid labelings of the unlink occurring in framing $(0, 0) \in (\mathbb{Z}_2)^2$ and those of the Hopf link occurring in framing $(1, 1) \in (\mathbb{Z}_2)^2$.



In example 1, the integral counting invariants defined by the given rack X do not distinguish the two links, but we can define an *enhancement*, i.e. a stronger invariant with the original invariant as a specialization, which does (see[10]):

Definition 3 Let X be a rack with rack rank N , L a link of c components, $W = (\mathbb{Z}_N)^c$ and for $\mathbf{w} = (w_1, \dots, w_c) \in W$ let $q^{\mathbf{w}} = q_1^{w_1} \dots q_c^{w_c}$. Then the *writhe-enhanced rack counting invariant* of L defined by X is

$$\phi_X^W(L) = \sum_{\mathbf{w} \in W} |\text{Hom}(FR(D(L, \mathbf{w})), X)| q^{\mathbf{w}}.$$

Example 2 In example 1, we had $\Phi_X^{\mathbb{Z}}(U_2) = 4 = \Phi_X^{\mathbb{Z}}(H_2)$; the writhe-enhanced invariant detects the difference, with $\Phi_X^W(U_2) = 4 \neq 4q_1q_2 = \Phi_X^W(H_2)$.

3 (t, s) -racks

We will now focus on a particular type of rack described in [7]. Let $\ddot{\Lambda} = \mathbb{Z}[t^{\pm 1}, s]/(s^2 - (1-t)s)$ and similarly let $\ddot{\Lambda}_n = \mathbb{Z}[t^{\pm 1}, s]/(n, s^2 - (1-t)s) = \mathbb{Z}_n[t^{\pm 1}, s]/(s^2 - (1-t)s)$.

Definition 4 Let X be a module over $\ddot{\Lambda}$. Then X is a rack with

$$x \triangleright y = tx + sy, \quad \text{and} \quad x \triangleright^{-1} y = t^{-1}(x - sy)$$

known as a (t, s) -rack [7]. If $s = 1 - t$ then X is a quandle known as an *Alexander quandle*.

Lemma 5 If X is a (t, s) -rack, then $\pi(x) = (t + s)x$.

Proof. Let X be a (t, s) -rack. Then for any $x \in X$, we have

$$\pi(x) = x \triangleright x = tx + sx = (t + s)x.$$

□

Corollary 6 For any (t, s) -rack X , the rack rank $N(X)$ is the minimal integer $N \geq 1$ such that $(t + s)^N x = x$ for all $x \in X$.

Let $X = R$ for a commutative ring R . We can make X a (t, s) -rack by selecting an invertible $t \in R$ and an element $s \in R$ satisfying $s^2 = (1 - t)s$. If R is finite, e.g. $R = \mathbb{Z}_n$, then X is a finite rack. Racks of this type with $R = \mathbb{Z}_n$ will be called *linear (t, s) -racks*, since we have $R = \ddot{\Lambda}_n/(t - a, s - b)$ for some $t = a, s = b \in \mathbb{Z}_n$. If R is a field, then either $s = 0$ and we have a constant action rack with $\sigma(x) = tx$, or s is invertible; if s is invertible, then $s^2 = (1 - t)s$ implies $s = 1 - t$ and our rack is a quandle. Thus, we have:

Proposition 7 Every linear (t, s) -rack $X = \mathbb{Z}_p$ for p prime is either a constant action rack or a linear Alexander quandle.

Example 3 The smallest nonquandle example of a linear (t, s) -rack is $X = \mathbb{Z}_4 = \{1, 2, 3, 4\}$ with $t = 1$ and $s = 2$. Then we have $s^2 = 2^2 = 4 = 0$ and $(1 - t)s = (1 - 1)2 = 0$. Since the kink map here is $\pi(x) = (s + t)x = 3x$ we have rack rank $N = 2$ and X is a non-quandle rack. The rack matrix of this rack is

$$M_X = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}.$$

Another way to get finite (t, s) -racks is to take quotients of $\ddot{\Lambda}$. The relation $s^2 = (1 - t)s$ says we can replace any power of s greater than 1 with an equivalent expression which is linear in s ; thus as an abelian group, we have $\ddot{\Lambda} \cong \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}[t^{\pm 1}]$. Then we can get finite (t, s) -racks by taking $\ddot{\Lambda}_n/(p(t))$ for a monic polynomial $p(t)$.

Example 4 Let $Y = \ddot{\Lambda}_2/(t + 1)$. The elements of Y include $0, 1, s$ and $1 + s$, and X has operation table

\triangleright	0	1	s	$1 + s$
0	0	s	0	s
1	1	$1 + s$	1	$1 + s$
s	s	0	s	0
$1 + s$	$1 + s$	1	$1 + s$	1

Thus, we appear to have a second example of a non-quandle (t, s) -rack of four elements. However, it is easy to check that this rack is isomorphic to the 4-element linear (t, s) -rack in example 3, via for example $\phi : Y \rightarrow X$ given by $\phi(0) = 4, \phi(1) = 1, \phi(s) = 2$ and $\phi(1 + s) = 3$. X and Y are not isomorphic as $\ddot{\Lambda}$ -modules, however, since their additive structures are different – as abelian groups, $X = \mathbb{Z}_4$ while $Y = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We can define a (t, s) -rack structure on any abelian group A by selecting an automorphism $t : A \rightarrow A$ and an endomorphism $s : A \rightarrow A$ satisfying the conditions that $st = ts$ and that $s^2 = (\text{Id} - t)s$.

Example 5 The linear (t, s) -rack in example 4 can be expressed as $X = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $t = \text{Id}$ and $s(x, y) = (0, x)$, while the linear (t, s) -rack in example 3 has $X = \mathbb{Z}_4$ with $t(x) = x$ and $s(x) = 2x$.

If $\phi : X \rightarrow Y$ is an isomorphism of $\check{\Lambda}$ -modules, then ϕ is also an isomorphism of (t, s) -racks; however, it is clear from examples 3 and 4 that rack isomorphism type does not determine $\check{\Lambda}$ -module structure. What conditions on $\check{\Lambda}$ -modules result in isomorphic (t, s) -racks? In [1, 9] we have a theorem about Alexander quandles, namely:

Theorem 8 *Two finite Alexander quandles M and M' are isomorphic as quandles iff*

- (i) $|M| = |M'|$ and
- (ii) *There exists a $\mathbb{Z}[t^{\pm 1}]$ -module isomorphism $h : (1 - t)M \rightarrow (1 - t)M'$.*

More colloquially, theorem 1 says that two Alexander quandles of the same finite cardinality are isomorphic iff their $(1 - t)$ -submodules $(1 - t)M$ and $(1 - t)M'$ are isomorphic as $\mathbb{Z}[t^{\pm 1}]$ -modules. We would like to generalize this result to (t, s) -racks. We first note that the straightforward generalization obtained by simply replacing $1 - t$ with s does not work; $X = \check{\Lambda}_4/(t - 1, s - 2)$ and $Y = \check{\Lambda}_4/(t - 3, s - 2)$ both have s -submodules sX and sY isomorphic to $\check{\Lambda}_2/(t - 1, s - 0)$ and $|X| = |Y|$, but Y is a quandle while X is a rack with rack rank 2.

As in the case of Alexander biquandles in [11], we are able to give necessary and sufficient conditions for two (t, s) -racks to be isomorphic. We first need a few lemmas:

Lemma 9 *If $\phi : X \rightarrow Y$ is a homomorphism of (t, s) -racks, then $\phi((t + s)x) = (t + s)\phi(x)$ for all $x \in X$.*

Proof.

$$\phi((t + s)x) = \phi(x \triangleright x) = \phi(x) \triangleright \phi(x) = (t + s)\phi(x).$$

□

Lemma 10 *Let X be a (t, s) -rack and let $z \in X$. The bijective map $p_z : X \rightarrow X$ defined by $p_z(x) = x + z$ is a rack isomorphism if and only if $\pi(z) = z$.*

Proof. Let X be a (t, x) -rack. Then for any $x, y, z \in X$ we have

$$p_z(x \triangleright y) = p_z(tx + sy) = tx + sy + z$$

while

$$p_z(x) \triangleright p_z(y) = (x + z) \triangleright (y + z) = tx + tz + sy + sz = tx + sy + (t + s)z.$$

Then $p_z(x \triangleright y) = p_z(x) \triangleright p_z(y)$ iff $z = (t + s)z = \pi(z)$. □

Let X be a (t, s) -rack and $A \subset X$ a subset. The $(t + s)$ -orbit of A , denoted $\mathcal{O}_{(t+s)}(A)$, is the set

$$\mathcal{O}_{(t+s)}(A) = \{(t + s)^k \alpha \mid \alpha \in A, k \in \mathbb{Z}\}.$$

We will be interested in the case where A is a set of coset representatives of X/sX ; note that in such a case multiple elements of $\mathcal{O}_{(t+s)}(A)$ may belong to the same coset of X/sX . Moreover, note that since $(t + s)$ is invertible, every element $x \in X$ can be written as $x = (t + s)y$ for some $y = \alpha + \omega$ with $\alpha \in A$, $\omega \in sX$; then we have

$$x = (t + s)\alpha + (t + s)\omega = (t + s)\alpha + \omega'$$

where $\alpha \in A$, $\omega' \in sX$. In particular, every element of $\mathcal{O}_{(t+s)}(A)$ can be written as $(t + s)\alpha + \omega$ for some $\alpha \in A$, $\omega \in sX$.

Theorem 11 *Two (t, s) -racks X, Y are isomorphic if and only if*

- (i) *There is an isomorphism of $\ddot{\Lambda}$ -submodules $h : sX \rightarrow sY$ and*
- (ii) *There are sets of coset representatives A, B for X/sX and Y/sY and a bijection*

$$g : \mathcal{O}_{(t+s)}(A) \rightarrow \mathcal{O}_{(t+s)}(B)$$

such that

$$h(s\alpha) = sg(\alpha)$$

for all $\alpha \in A$ and

$$g((t+s)\alpha + \omega) = (t+s)g(\alpha) + h(\omega)$$

for all $(t+s)\alpha + \omega \in \mathcal{O}_{(t+s)}(A)$ with $\alpha \in A$ and $\omega \in sX$.

Proof.

(\Rightarrow) Let $\phi : X \rightarrow Y$ be an isomorphism of (t, s) -racks. In X we have $0 \triangleright 0 = t0 + s0 = 0 + 0 = 0$ so $N(0) = 1$, and lemma 2 implies $N(\phi(0)) = 1$. Then by lemma 10 we may assume without loss of generality that $\phi(0) = 0$ since if not, we can replace ϕ with $p_{-\phi(0)} \circ \phi$.

Since ϕ is a (t, s) -rack homomorphism we have

$$\phi(tx + sy) = \phi(x \triangleright y) = \phi(x) \triangleright \phi(y) = t\phi(x) + s\phi(y)$$

and since $\phi(0) = 0$ we have

$$\phi(tx) = \phi(tx + s0) = t\phi(x) + s\phi(0) = tx + s0 = tx$$

and

$$\phi(sy) = \phi(t0 + sy) = t\phi(0) + s\phi(y) = t0 + sy = sy.$$

Since t is invertible, every element $x \in X$ is $t(t^{-1}x)$. Then we have

$$\begin{aligned} \phi(sx + sy) &= \phi(t(t^{-1}sx) + sy) \\ &= t\phi(t^{-1}sx) + s\phi(y) \\ &= \phi(tt^{-1}sx) + \phi(sy) \\ &= \phi(sx) + \phi(sy). \end{aligned}$$

Not every $x \in X$ need satisfy $x = sz$ for some $z \in X$, but for those that do, i.e. for the submodule sX , we have ϕ preserving multiplication by both t and s and preserving addition, so the restriction $h = \phi|_{sX}$ is an isomorphism of $\ddot{\Lambda}$ -modules.

Now, let A be any set of coset representatives of X/sX . Define $g = \phi|_{\mathcal{O}_{(t+s)}(A)}$ and set $B = \{\phi(\alpha) \mid \alpha \in A\}$. Then for each $\alpha \in A$ we have

$$h(s\alpha) = \phi(s\alpha) = s\phi(\alpha) = sg(\alpha)$$

and for any $(t+s)\alpha + \omega \in \mathcal{O}_{(t+s)}(A)$ with $\omega = s\gamma$ we have

$$\begin{aligned} g((t+s)\alpha + \omega) &= \phi((t+s)\alpha + \omega) = \phi(tt^{-1}(t+s)\alpha + s\gamma) \\ &= t\phi(t^{-1}(t+s)\alpha) + s\phi(\gamma) \\ &= \phi(tt^{-1}(t+s)\alpha) + \phi(s\gamma) \\ &= (t+s)\phi(\alpha) + \phi(\omega) \\ &= (t+s)g(\alpha) + h(\omega) \end{aligned}$$

as required.

Finally, note that B is a set of coset representatives for Y/sY since if $\beta - \beta' \in sY$ for any $\beta, \beta' \in B$ then $\beta = \beta' + s\gamma$ and we have

$$\phi^{-1}(\beta) = \phi^{-1}(t\phi^{-1}(\beta') + s\gamma) = t\phi^{-1}(\beta') + s\phi^{-1}(\gamma) = \phi^{-1}(t\phi^{-1}(\beta)) + \phi^{-1}(s\gamma) = \phi^{-1}(\beta') + \phi^{-1}(s\gamma)$$

and the corresponding $\alpha = \phi^{-1}\beta$, $\alpha' = \phi^{-1}(\beta')$ satisfy $\alpha - \alpha' \in sX$.

(\Leftarrow) Let X and Y be (t, s) -racks, $h : sX \rightarrow sY$ an isomorphism of $\check{\Lambda}$ -modules, and suppose $A \subset X$ and $B \subset Y$ are sets of coset representatives of X/sX and Y/sY respectively, with a bijection $g : \mathcal{O}_{(t+s)}(A) \rightarrow \mathcal{O}_{(t+s)}(B)$ satisfying

$$h(s\alpha) = sg(\alpha)$$

for all $\alpha \in A$ and

$$g((t+s)\alpha + \omega) = (t+s)g(\alpha) + h(\omega)$$

for all $(t+s)\alpha + \omega \in \mathcal{O}_{(t+s)}(A)$. In particular, $\omega = 0$ says $g((t+s)^k\alpha) = (t+s)^k g(\alpha)$. Define $\phi : X \rightarrow Y$ by

$$\phi(\alpha + \omega) = g(\alpha) + h(\omega)$$

where $\alpha \in A$ and $\omega \in sX$.

To see that ϕ is well-defined, suppose $\alpha + \omega = \alpha' + \omega'$ where $\alpha, \alpha' \in \mathcal{O}_{(t+s)}(A)$ and $\omega, \omega' \in sX$. Then $\alpha' = \alpha + (\omega - \omega')$ and we have

$$\begin{aligned} \phi(\alpha' + \omega') &= g(\alpha') + h(\omega') \\ &= g(\alpha + (\omega - \omega')) + h(\omega') \\ &= g((t+s)(t+s)^{-1}\alpha + (\omega - \omega')) + h(\omega') \\ &= (t+s)g((t+s)^{-1}\alpha) + h(\omega - \omega') + h(\omega') \\ &= (t+s)g((t+s)^{-1}\alpha) + h(\omega) - h(\omega') + h(\omega') \\ &= (t+s)g((t+s)^{-1}\alpha) + h(\omega) \\ &= g(\alpha) + h(\omega) \\ &= \phi(\alpha + \omega) \end{aligned}$$

To see that ϕ is bijective, note that we can define $\phi^{-1} : Y \rightarrow X$ by $\phi^{-1}(\beta + \gamma) = g^{-1}(\beta) + h^{-1}(\gamma)$.

To see that ϕ is a homomorphism of (t, s) -racks, let $x = \alpha + \omega$ and $y = \alpha' + \omega'$ with $\alpha, \alpha' \in A$ and $\omega, \omega' \in sX$, and note that $t\alpha = (t+s)\alpha - s\alpha$. Then

$$\begin{aligned} \phi(x \triangleright y) &= \phi(t(\alpha + \omega) + s(\alpha' + \omega')) \\ &= \phi((t+s)\alpha - s\alpha + t\omega + s\alpha' + s\omega') \\ &= g((t+s)\alpha) + h(-s\alpha + t\omega + s\alpha' + s\omega') \\ &= (t+s)g(\alpha) - h(s\alpha) + th(\omega) + h(s\alpha') + sh(\omega') \\ &= tg(\alpha) + sg(\alpha) - h(s\alpha) + th(\omega) + h(s\alpha') + sh(\omega') \\ &= tg(\alpha) + h(s\alpha) - h(s\alpha) + th(\omega) + sg(\alpha') + sh(\omega') \\ &= tg(\alpha) + th(\omega) + sg(\alpha') + sh(\omega') \\ &= t(g(\alpha) + h(\omega)) + s(g(\alpha') + h(\omega')) \\ &= \phi(x) \triangleright \phi(y) \end{aligned}$$

as required. □

Remark 1 Note that if $s = 1 - t$ and X is an Alexander quandle, then $s + t = 1$ and $\mathcal{O}_{s+t}(A)$ is just A . Then the fact that A is a set of coset representatives means that condition (ii) reduces to the requirement that $h(s\alpha) = sg(\alpha)$, and it is shown in [10] that this condition can always be satisfied.

We end this section with a few interesting observations about (t, s) -racks.

In every rack, the \triangleright and \triangleright^{-1} operations are right-distributive¹; if a quandle is Alexander, however, the quandle operations are also left-distributive. This property does not extend to more general (t, s) -racks:

Proposition 12 *A (t, s) -rack X is left-distributive if and only if X is an Alexander quandle.*

Proof. Let X be a (t, s) -rack. Then

$$x \triangleright (y \triangleright z) = tx + s(ty + sz) = tx + tsy + s^2z$$

while

$$(x \triangleright y) \triangleright (x \triangleright z) = t(tx + sy) + s(tx + sz) = (t^2 + st)x + tsy + s^2z.$$

Then $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ if and only if $(ts + t^2)x = tx$ for all $x \in X$, i.e., iff $t^2x = t(1 - s)x$, which implies $tx = (1 - s)x$ and hence $sx = (1 - t)x$. We then have a $\mathbb{Z}[t^{\pm 1}]$ -module structure on X induced by taking the quotient of \tilde{A} by the ideal generated by $s - (1 - t)$, and the (t, s) -rack operation on X becomes

$$x \triangleright y = tx + sy = tx + (1 - t)y$$

and X is an Alexander quandle. □

Our next observation notes that (t, s) -racks contain Alexander quandles not just in the categorical sense, but literally:

Proposition 13 *Let X be a rack. The subset $Q(X) \subset X$ of all elements of X of rack rank $N = 1$ is a quandle, known as the maximal subquandle of X . If X is a (t, s) -rack, then $Q(X)$ is an Alexander quandle.*

Proof. To see that $Q(X) = \{x \in X \mid x \triangleright x = x\}$ is a subrack, note that $x, y \in Q$ implies

$$x \triangleright y = (x \triangleright x) \triangleright y = (x \triangleright y) \triangleright (x \triangleright y)$$

and $x \triangleright y \in Q$. Then $Q(X)$ is a rack with rack rank $N = 1$, so $Q(X)$ is a quandle.

Now let X be a (t, s) -rack. To see that Q is Alexander, note that if $x \in Q$ then $(t + s)x = x$ and we have $sx = (1 - t)x$. Then for any $x, y \in Q$, we have

$$x \triangleright y = tx + sy = tx + (1 - t)y$$

and Q is an Alexander quandle. □

For general racks $Q(X)$ may be empty (none of the quandle axioms are existentially quantified, so the empty set satisfies the quandle axioms vacuously), but for (t, s) -racks $Q(X)$ always contains at least 0. Indeed, we have

Corollary 14 *For any (t, s) -rack X , sX is a subquandle of $Q(X)$.*

Proof. To see that sX is closed under \triangleright , note that

$$sx \triangleright sy = tsx + s^2y = s(tx + sy) \in sX.$$

To see that $sX \subset Q(X)$, let $x = sx'$; then we have

$$sx' \triangleright sx' = tsx' + s^2x' = (ts + s^2)x' = sx'$$

and $x \in Q(X)$. □

We note that sX may be a proper subquandle of $Q(X)$: take for instance $X = \mathbb{Z}_4$ with $t = 3$ and $s = 2$; then $t + s = 1$ and $Q(X) = X$, but $sX = \{0, 2\} \subsetneq X$.

¹At least, when we write the quandle operation as a right action following Joyce. In some works such as [1] the rack operations are written as left actions, in which case the rack axioms require left-distributivity.

4 Enhanced link invariants

In this section we define a few enhancements of the rack counting invariant $\Phi_X^{\mathbb{Z}}$ when X is a (t, s) -rack.

For our first enhancement, we note that a (t, s) -rack is not just a rack, but also has the structure of a $\ddot{\Lambda}$ -module. We can use this extra structure to define enhancements of the rack counting invariant. Let T be a finite (t, s) -rack with rack rank N and let A_T be T considered as an abelian group. For any subset $S \subset T$, let $AC(S)$ be the additive closure of S , i.e. the subgroup of A_T generated by S . For each homomorphism $f : FR(L, \mathbf{w}) \rightarrow T$ we can use the additive closure of the image subrack of f , as a signature of f to obtain an enhancement of $\Phi_X^{\mathbb{Z}}$.

Definition 5 Let X be a (t, s) -rack and $L = L_1 \cup \dots \cup L_c$ an oriented link of c ordered components. The *additive (t, s) -rack enhanced multiset* of L with respect to T is the multiset of abelian groups

$$\Phi_X^{ts,+M}(L) = \{AC(\text{Im}(f)) \mid f \in \text{Hom}(FR(L, \mathbf{w}), X), \mathbf{w} \in W\}.$$

For ease of comparison, we also define the *additive (t, s) -rack enhanced polynomial* of L with respect to X to be

$$\Phi_X^{ts,+}(L) = \sum_{\mathbf{w} \in W} \left(\sum_{f \in \text{Hom}(FR(L, \mathbf{w}), X)} u^{|\text{Im}(f)|} \right)$$

where $W = (\mathbb{Z}_N)^c$.

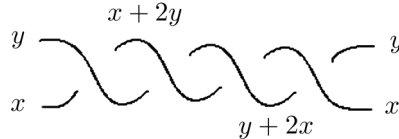
Note that for any $f \in \text{Hom}(FR(L, \mathbf{w}), X)$, X -labeled blackboard-framed Reidemeister moves and N -phone cord moves do not change the image subrack $\text{Im}(f)$, and thus the above quantities are link invariants. It is clear that the rack counting invariant can be obtained from either form of the enhanced invariant by taking the cardinality in the multiset case or by evaluating $u = 1$ in the polynomial case. We also note that the multiset form of the invariant is stronger than the polynomial form since the polynomial form forgets the abelian group structure of the signatures, keeping only their cardinalities.

In the proof of proposition 15 we illustrate a method for computing $\Phi_X^{ts,+}$ by computing $\Phi_X^{ts,+}(L)$ for all $(2, n)$ -torus links for a choice of (t, s) -rack X .

Proposition 15 Let $X = \mathbb{Z}_4$ with (t, s) -rack operation $x \triangleright y = x + 2y$. Then the $(2, n)$ torus link $T_{(2,n)}$ has $\Phi_X^{ts,+}$ values given by

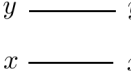
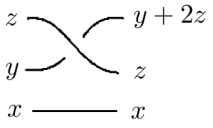
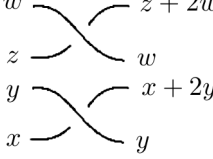
$$\Phi_X^{ts,+}(T_{(2,n)}) = \begin{cases} 4u + 12u^2 + 20u^4, & n \equiv 0 \pmod{4}, \\ 2u + 2u^2 + 2u^4, & n \equiv 1, 3 \pmod{4}, \\ 4u + 12u^2 + 4u^4, & n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Let X be the (t, s) -rack $X = \mathbb{Z}_4$ with $t = 1$ and $s = 2$. Recall that the $(2, n)$ torus link $T_{(2,n)}$ is the closure of the 2-strand braid with n positive twists. We first note that the set of rack labelings of $T_{(2,n)}$ is periodic with period 4:





There are four cases; we will show the first two. Closing the braid gives us a system of equations in \mathbb{Z}_4 ; each solution to this system determines a valid rack labeling.

When $n \equiv 0 \pmod{4}$, $T_{(2,n)}$ is a two-component link and we need to stabilize once on each component (the braid version of the Reidemeister I move) in order to get a complete set of writhes mod 2:

Writhe vector	Diagram	System	Reduces to	Contribution
$(0,0)$		$y = y$ $x = x$	$y = y$ $x = x$	$u + 3u^2 + 12u^4$
$(0,1)$		$z = y + 2z$ $y = z$ $x = x$	$y = 3y$ $x = x$	$u + 3u^2 + 4u^4$
$(1,1)$		$w = z + 2w$ $z = w$ $y = x + 2y$ $x = y$	$z = 3z$ $x = 3x$	$u + 3u^2$

The writhe vector $(1,0)$ has the same contribution as the $(0,1)$ writhe vector due to the symmetry of the link. Thus, we have $\Phi_X^{ts,+}(T_{(2,n)}) = 4u + 12u^2 + 20u^4$ for $n \equiv 0 \pmod{4}$.

When $n \equiv 1 \pmod{4}$, $T_{(2,n)}$ is a knot and we need to stabilize once:

Writhe	Diagram	System	Reduces to	Contribution
0		$y = x + 2y$ $x = y$	$2x = 0$ $x = y$	$u + u^2$
1		$z = x + 2y + 2z$ $y = z$ $x = y$	$x = y = z$	$u + u^2 + 2u^4$

Thus, we have $\Phi_X^{ts,+}(T_{(2,n)}) = 2u + 2u^2 + 2u^4$ for $n \equiv 1 \pmod{4}$.

Similar computations give us the $n \equiv 2, 3 \pmod{4}$ cases. □

Remark 2 Repeating the computation in proposition 15 with the (t, s) -rack structure $X = \ddot{\Lambda}_2/(t^2 + 1)$, we get for example

$$\Phi_X^{ts,+}(T_{(2,n)}) = 4u + 22u^2 + 10u^4$$

when $n \equiv 0 \pmod{4}$. This result shows that different extra structures on the same rack can indeed define different enhanced invariants, as suggested in [8]. While this may seem counter-intuitive, it merely reflects the fact that different extra structures on a rack (or quandle, biquandle, etc.) differ in which labelings get assigned different signatures. In particular, to define $\Phi_X^{ts,+}$ it is not enough to know the rack matrix of X ; we need the full $\ddot{\Lambda}$ -module structure.

Our next example gives a table of values of $\Phi_X^{ts,+}(L)$ for all prime knots with up to eight crossings and all prime links with up to seven crossings using our `python` code, available at <http://www.esotericka.org>. In particular, these values demonstrate that $\Phi_X^{ts,+}(L)$ is stronger in general than the integral counting invariant $\Phi_X^{\mathbb{Z}}(L)$.

Example 6 Let X be the (t, s) -rack $X = \mathbb{Z}_{12}$ with $t = 11$ and $s = 2$; in fact, X is an Alexander quandle. We computed the value of $\Phi_X^{ts,+}$ on all prime knots with up to eight crossings and all prime links with up to seven crossings as listed in the Knot Atlas [2] using `python` code available at <http://www.esotericka.org>. The results are collected in the table below. Note in particular that while $\Phi_X^{ts,+}$ determines $\Phi_X^{\mathbb{Z}}$ (since $\Phi_X^{\mathbb{Z}}$ may be obtained from $\Phi_X^{ts,+}$ by setting $u = 1$), the values in this example demonstrate that $\Phi_X^{\mathbb{Z}}$ does not determine $\Phi_X^{ts,+}$ – for example, both $L6a1$ and $L6a5$ have $\Phi_X^{\mathbb{Z}} = 144$, but are distinguished by $\Phi_X^{ts,+}$ – making $\Phi_X^{ts,+}$ a strictly stronger invariant.

$\Phi_X^{ts,+}(L)$	L
$u + u^2 + 2u^3 + 2u^4 + 2u^6 + 4u^{12}$	$4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9,$ $8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17},$
$u + u^2 + 8u^3 + 2u^4 + 8u^6 + 16u^{12}$	$3_1, 6_1, 7_4, 7_7, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21}$
$u + u^2 + 26u^3 + 2u^4 + 26u^6 + 52u^{12}$	8_{18}
$u + 3u^2 + 2u^3 + 4u^4 + 6u^6 + 8u^{12}$	$L2a1, L6a2, L7a6$
$u + 3u^2 + 2u^3 + 12u^4 + 6u^6 + 24u^{12}$	$L4a1, L5a1, L7a2, L7a3, L7a4, L7n1, L7n2$
$u + 7u^2 + 2u^3 + 8u^4 + 14u^6 + 16u^{12}$	$L6n1, L7a7$
$u + 3u^2 + 8u^3 + 4u^4 + 24u^6 + 32u^{12}$	$L6a3, L7a5$
$u + 7u^2 + 8u^3 + 8u^4 + 56u^6 + 64u^{12}$	$L6a5$
$u + 3u^2 + 8u^3 + 12u^4 + 24u^6 + 96u^{12}$	$L6a1, L7a1$
$u + 7u^2 + 2u^3 + 56u^4 + 14u^6 + 112u^{12}$	$L6a4$

For our next enhancement, we note that multiplication by s in a (t, s) -rack is a rack homomorphism which projects X onto the Alexander subrack sX ; if $z = x \triangleright y = tx + sy$ then we have

$$sx \triangleright sy = t(sx) + s(sy) = s(tx + sy) = sz$$

In particular, for any X -labeling f of a link diagram, there is a corresponding sX -labeling obtained by multiplying every label by s . Since this corresponding labeling is preserved by blackboard-framed Reidemeister moves and N -phone cord moves, the projection onto the subrack $s\text{Im}(f) = \text{Im}(s(f))$ can be used as a signature of f to define enhancements.

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \text{\scriptsize } z \end{array} & \begin{array}{c} \uparrow \\ \text{\scriptsize } y \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \text{\scriptsize } x \end{array} \\
 & & \xrightarrow{\times s} \\
 & & \begin{array}{c} \xrightarrow{\quad} \\ \text{\scriptsize } sz \end{array} \\
 & \begin{array}{c} \uparrow \\ \text{\scriptsize } sy \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \text{\scriptsize } sx \end{array}
 \end{array}$$

Thus we have:

Definition 6 Let X be a (t, s) -rack with rack rank N , $L = L_1 \cup \dots \cup L_c$ an oriented link of c ordered components, and $W = (\mathbb{Z}_N)^c$ the space of writhe vectors mod N . For each rack homomorphism $g : FR(L, \mathbf{w}) \rightarrow sX$, let

$$s^{-1}(g) = \{f \in \text{Hom}(FR(L, \mathbf{w}), X) : g = sf\}$$

be the set of rack labelings of (L, \mathbf{w}) by X which project to g under multiplication by s . Then the s -enhanced multiset of L with respect to X is

$$\Phi_X^{ts,sM}(L) = \{s^{-1}(g) : g \in \text{Hom}(FR(L, \mathbf{w}), sX), \mathbf{w} \in W\}$$

and the s -enhanced polynomial of L with respect to X is

$$\Phi_X^{ts,s}(L) = \sum_{\mathbf{w} \in W} \left(\sum_{g \in \text{Hom}(FR(L, \mathbf{w}), sX)} u^{|s^{-1}(g)|} \right).$$

Remark 3 This enhancement is slightly different from the usual enhancements in that recovery of $\Phi_X^{\mathbb{Z}}$ requires not evaluating $\Phi_X^{ts,s}$ at $u = 1$ but rather summing the product of the coefficient times exponent for each term in $\Phi_X^{ts,s}$. Note that we can also regard $\Phi_X^{ts,s}$ as an enhancement of the quandle counting invariant with respect to the Alexander subquandle sX ; this enhancement is related to the rack module enhancements defined in [3].

Example 7 Taking the (t, s) -rack $X = \mathbb{Z}_4$ with $t = 3$ and $s = 2$, we computed $\Phi_X^{ts,s}$ for all prime links with up to seven crossings as listed in the Knot Atlas [2]. The results are listed in the table below; in particular, note that $\Phi_X^{ts,s}$ distinguishes the links $L4a1$ and $L6a5$ which have the same integral rack counting invariant value $\Phi_X^{\mathbb{Z}} = 16$. Since this rack has orbits which are constant action racks, the invariant has the same value, $\Phi_X^{ts,s} = 2u^2$, for all knots.

$\Phi_X^{ts,s}$	L
$2u + 2u^3$	$L2a1, L6a2, L6a3, L7a5, L7a6$
$2u + 2u^2 + 2u^5$	$L6a5, L6n1, L7a7$
$4u + 4u^3$	$L4a1, L5a1, L6a1, L7a1, L7a2, L7a3, L7a4, L7n1, L7n2$
$8u + 8u^2 + 8u^5$	$L6a4$

We end this section with an application. In recent works such as [12, 4], a partial ordering on knot types is defined by setting

$$K > K' \iff \exists \phi : \pi_1(S^3 \setminus K) \rightarrow \pi_1(S^3 \setminus K')$$

where ϕ is a surjective group homomorphism. Replacing the knot group with the knot quandle yields a related ordering in which ϕ is required to preserve peripheral structure.

Let us define a partial ordering \succ on $\mathbb{Z}[u]$ by

$$\sum_{k=0}^n \alpha_k u^k \succ \sum_{k=0}^n \beta_k u^k \iff \alpha_k > \beta_k$$

for all $k = 0, \dots, n$. Then we have:

Proposition 16 *If there exists a surjective homomorphism from the knot quandle of a knot K onto the knot quandle of K' , then*

$$\Phi_X^{ts,+}(K) \succ \Phi_X^{ts,+}(K')$$

for all Alexander quandles X .

Proof. For any quandle homomorphism $f : Q(K') \rightarrow X$, the map $f \circ \phi : Q(K) \rightarrow X$ is a quandle homomorphism. Moreover, $\text{Im}(f) \subset \text{Im}(f \circ \phi)$, since $x \in \text{Im}(f)$ says $x = f(a)$ for some $a \in Q(K')$, and surjectivity of ϕ then says $a = \phi(b)$ for some $b \in Q(K)$; then we have $x = f(a) = f(\phi(b)) = f \circ \phi(b)$ and $x \in \text{Im}(f \circ \phi)$. Conversely, if $x \in \text{Im}(f \circ \phi)$ then $x = f \circ \phi(b)$ for some $b \in Q(X)$ and $x = f(\phi(b))$ implies $x \in \text{Im}(f)$. Thus we have $\text{Im}(f) = \text{Im}(f \circ \phi)$.

Then every contribution $u^{|AC(\text{Im}(f))|}$ to $\Phi_X^{ts,+}(K')$ is matched by an equal contribution to $\Phi_X^{ts,+}(K)$, and we have

$$\Phi_X^{ts,+}(K) \succ \Phi_X^{ts,+}(K')$$

as required. □

This proposition means that for Alexander quandles X , $\Phi_X^{ts,+}(K)$ can provide us with obstructions for knot ordering. Indeed, every finite Alexander quandle X defines its own partial ordering $>_X$ of knots by

$$K >_X K' \iff \Phi_X^{ts,+}(K) \succ \Phi_X^{ts,+}(K').$$

For instance, in the quandle ordering defined by the Alexander quandle $X = \check{\mathbb{A}}_{12}/(t - 11, s - 2)$ in example 6, we have $4_1 <_X 3_1 <_X 8_{18}$, etc.

5 Questions

In this section we collect questions for future research.

Let X be a (t, s) -rack. When $(t + s)x = x$ so that X is an Alexander quandle, the enhanced invariants defined in section 4 are also defined for knotted surfaces in \mathbb{R}^4 . We have not looked in any detail at how effective these enhancements may be at distinguishing knotted surfaces or what relationship they might have with triple point number, etc. This might prove to be an interesting direction for future investigation.

The enhanced invariants defined in section 4 are also well-defined without modification for virtual knots and links. It is known that certain writhe-enhanced rack counting invariant values are impossible for classical links but possible for virtual links, providing a method of detecting non-classicality. Does anything similar happen with (t, s) -rack enhanced invariants?

The conditions given in proposition 11 seem unsatisfying; is there a simpler necessary and sufficient condition which can replace (ii), e.g., X and Y have conjugate kink maps in $S_{|X|}$ or equal rack polynomials?

In light of the observations at the end of section 3, a (t, s) -rack with rack rank N of the form \tilde{A}/I for an ideal I can be viewed as an extension of an Alexander quandle $A = \mathbb{Z}[t^{\pm 1}]/I'$ obtained by adjoining a variable s and modding out by $s^2 - (1 - t)s$, $(s + t)^N - 1$ and possibly additional polynomials. What conditions on these polynomials are required to yield isomorphic (t, s) -racks?

References

- [1] N. Andruskiewitsch and M. Graña. From racks to pointed Hopf algebras. *Adv. Math.* **178** (2003) 177-243.
- [2] D. Bar-Natan (Ed.). The Knot Atlas. http://katlas.math.toronto.edu/wiki/Main_Page.
- [3] A. Haas, G. Hackel, S. Nelson, J. Yuen and Q. Zhang. Rack module enhancements of counting invariants. arXiv:1008.0114.
- [4] S. M. Garraabrant, J. Hoste and P. D. Shanahan. Upper bounds in the Ohtsuki-Riley-Sakuma partial order on 2-bridge knots. arXiv:1007.3278.
- [5] D. Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra* **23** (1982) 37-65.
- [6] S. V. Matveev. Distributive groupoids in knot theory. *Math. USSR, Sb.* **47** (1984) 73-83.
- [7] R. Fenn and C. Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications* **1** (1992) 343-406.
- [8] E.A. Navas and S. Nelson. On symplectic quandles. *Osaka J. Math.* **45** (2008) 973-985.
- [9] S. Nelson. Classification of finite Alexander quandles. *Topol. Proc.* **27** (2003) 245-258.
- [10] S. Nelson. Link invariants from finite racks. arXiv:0808.0029.
- [11] D. Lam and S. Nelson. An isomorphism theorem for Alexander biquandles. *Intl. J. Math.* **20** (2009) 97-107.
- [12] T. Ohtsuki, R. Riley and M. Sakuma. Epimorphisms between 2-bridge link groups. The Zieschang Gedenkschrift, *Geom. Topol. Monogr.* **14** Geom. Topol. Publ., Coventry, (2008) 417-450.

DEPARTMENT OF MATHEMATICAL SCIENCES,
CLAREMONT MCKENNA COLLEGE,
850 COLUMBIA AVE.,
CLAREMONT, CA 91711